Given name:____________________ **Family name:**____________________

Student number:__________________ **Signature:**____________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

MAT 334 Y (Complex Variables)

Instructor: Yuri Burda

SPECIAL DEFERRED EXAMINATION
September 23, 2011

Duration: 3 hours

No aids allowed

This examination paper consists of **8** pages and **5** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer all questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (20)	
2 (20)	
3 (20)	
4 (20)	
5 (20)	
Total (100)	

1. Evaluate the following integrals:

(a) [10]

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)^2}\, dx$$

(b) [10]

$$\int_{-\pi}^{\pi} \frac{d\theta}{5 - 3\sin\theta}$$

2. (a) [10] Give an example of a Möbius transformation that maps the outside of the unit circle to the inside of the unit circle.

(b) [10] Show that every function f analytic on the complex plane and satisfying $|f(z)| > 1$ for all z is constant.

3. Evaluate the following expressions:

(a) [5]

$$\int |z|^2 \, dz$$

over the straight line segment from $z = 1 + i$ to $z = 2 + 2i$

(b) [5]

$$\int \cos(i\pi z) \, dz$$

over the curve $z(t) = t^2 - t + it$ for $0 \leq t \leq 1$

(c) [5]

$$\int \frac{1}{\sin z}\, dz$$

over the circle $|z - i| = 3$

(d) [5]

$$\lim_{R\to\infty} \int_{C_R} \sin\left(\frac{1}{z}\right)\, dz$$

where C_R is the semicircle $z(t) = Re^{it}$ for $0 \leq t \leq \pi$ (hint: you can use Laurent series expansion of $\sin\left(\frac{1}{z}\right)$ in powers of z)

4. (a) [5] Find the number of zeroes of the function

$$f(z) = 2z^4 - iz^3 + z^2 + 4iz - 1$$

in the upper half-plane $\operatorname{Im} z > 0$.

(b) [5] Find the number of zeroes of the function $f(z) = z^{2011} + 4z^4 - e^z$ in the disc $|z| < 1$.

(c) [5] Find the order of the zero of the function

$$f(z) = (\cos z - 1)^2 (e^{iz} - 1)^3 \, (\tan z)^4$$

at $z = 2\pi$

(d) [5] Give an example of a non-constant analytic function $f(z)$ such that the equation $f(z) = 1$ doesn't have any solutions.

5. For each of the following statements determine whether it is true or false and write a short explanation supporting your answer (a guess without an explanation will not be graded).

(a) [4] The function $u(x,y) = x^2 - y^2$ is harmonic

(b) [4] The function $v(x,y) = \frac{x}{x^2+y^2}$ is a harmonic conjugate of $u(x,y) = \frac{y}{x^2+y^2}$ (i.e. $f(x+iy) = u(x,y)+iv(x,y)$ is analytic) in the domain $(x,y) \neq (0,0)$.

(c) [3] There exists an analytic (one-valued) function on the complex plane whose square is equal to $\sin(z)$
(hint: what could be its order of zero at $z = 0$?)

(d) [3] There exists an analytic (one-valued) function on the complex plane whose square is equal to $4\sin^2 z - 4$

(e) [3] All values of $(-1)^{2i}$ are real numbers

(f) [3] There are real numbers among the values of 2^{-i}