Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**____________________

UNIVERSITY OF TORONTO
Faculty of Arts and Science

MAT 334 Y (Complex Variables)
Instructor: Yuri Burda

MIDTERM TEST
June 30, 2011

Duration: 90 minutes

No aids allowed

This examination paper consists of **6** pages and **5** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer all questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (20)	
2 (20)	
3 (20)	
4 (20)	
5 (20)	
Total (100)	

1. (a) [10] Find all complex solutions z of the following equation (express them in the form $a+ib$):

$$(2/\bar{z}+1)^3 = 1$$

$$2/\bar{z}+1 = 1^{1/3} = e^{2\pi ik/3} = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$2/\bar{z} = 0, -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\bar{z} = 2/(-\frac{3}{2} \pm \frac{\sqrt{3}}{2}i)$$

$$\bar{z} = \frac{2(-\frac{3}{2} \mp \frac{\sqrt{3}}{2}i)}{9/4+3/4}$$

$$z = -1 \pm i/\sqrt{3}$$

(b) [10] Sketch the set of complex numbers satisfying $|2/\bar{z}+1|^3 = 1$

$$|2/\bar{z}+1| = 1$$

$$|2+\bar{z}| = |\bar{z}|$$

$$(2+x)^2 + y^2 = x^2 + y^2 \text{ where } z = x+iy$$

$$x = -1$$

The answer is the vertical line $x = -1$

Is this set open?
Closed?
Connected?
Does it contain points not in its boundary?

It is not open, it is closed, it is connected, it does not contain points not in its boundary

2. (a) [10] Find the radius of convergence of

$$\sum_{n=0}^{\infty} e^{(1+i)n} z^n$$

$$R = \liminf \frac{1}{|e^{(1+i)n}|^{1/n}} = \frac{1}{e}$$

(b) [10] Compute

$$\sum_{n=1}^{\infty} n \left(\frac{1+2i}{5} \right)^{n-1}$$

or show that the sum diverges.

The series is of the form $\sum nz^{n-1}$ for $z = \frac{1+2i}{5}$. This power series converges for $|z| < \liminf \frac{1}{n^{1/n}} = 1$ in particular for $z = \frac{1+2i}{5}$ ($|\frac{1+2i}{5}| = \frac{1}{\sqrt{5}} < 1$).
To find what the series is equal to we notice $\sum nz^n = (\sum z^n)' = (1/(1-z))' = 1/(1-z)^2$. In particular

$$\sum_{n=1}^{\infty} n \left(\frac{1+2i}{5} \right)^{n-1} = \frac{1}{\left(1 - \frac{1+2i}{5}\right)^2} = 3/4 + i$$

3. Evaluate the integral

$$\int_C \frac{e^{\pi z/2}}{z^2+4}dz$$

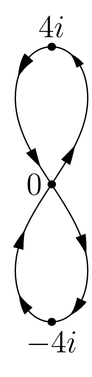
(a) [7] For the curve C being the circle $|z-3i| = 2$ oriented counter-clockwise.

By Cauchy formula $\int_C \frac{e^{\pi z/2}/(z+2i)}{z-2i}dz = 2\pi i e^{\pi 2i/2}/(2i+2i) = -\pi/2$

(b) [7] For the curve C being the circle $|z+3i| = 2$ oriented counter-clockwise.

By Cauchy formula $\int_C \frac{e^{\pi z/2}/(z-2i)}{z+2i}dz = 2\pi i e^{\pi(-2i)/2}/(-2i-2i) = \pi/2$

(c) [6] For the curve C in the following picture:



The curve goes once counter-clockwise around $z = 2i$ and once clockwise around $z = -2i$. Hence this integral is equal to the difference of answers in a) and b), i.e. $-\pi/2 - \pi/2 = \pi$.

4. (a) [10] Let

$$f(z) = \begin{cases} \frac{|z|^2-1}{z-1} & \text{if } z \neq 1 \\ 2 & \text{if } z = 1 \end{cases}$$

Determine whether the function $f(z)$ is continuous at the point $z = 1$ (explain why or why not).

It isn't continuous: if z approaches 1 along the path $z = 1 + it$, $t \to 0$, then $\frac{|z|^2-1}{z-1} = \frac{1+t^2-1}{it} = -it \to 0 \neq 2$

(b) [10] Let $g(z) = z^2 - |z|^2$. Determine all the points where $g'(z)$ exists. Find $g'(z)$ at each such point.

Verify Cauchy Riemann equations for $f(z) = z^2 - |z|^2 = -2y^2 + 2xyi$ where $z = x + iy$:

$$0 = 2x$$
$$4y = 2y$$

or $x = 0, y = 0$. Hence the only point where the derivative might exist is the origin. To compute the value of the derivative we follow the definition

$$f'(0) = \lim_{z \to 0} \frac{z^2 - |z|^2}{z} = \lim_{z \to 0}(z - \bar{z}) = 0$$

5. [20] Suppose that the function f is analytic (i.e. complex-differentiable) at all points of the complex plane. Suppose also that the image of f is contained in the unit circle $\{u + iv | u^2 + v^2 = 1\}$. Show that f is constant.

Differentiating $u^2 + v^2 = 1$ with respect to x and y we get $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Using Cauchy-Riemann equations we get

$$uv_y + vv_x = 0 \quad (1)$$
$$-uv_x + vv_y = 0 \quad (2)$$

Multiplying (1) by u and adding (2) $\times\, v$ we get $(u^2 + v^2)v_y = 0$, or, since $u^2 + v^2 = 1$, $v_y = 0$.

Similarly (1) $\times\, v$ − (2) $\times\, u$ gives $v_x = 0$.

Cauchy-Riemann equations then imply $u_x = 0, u_y = 0$ and hence both u and v are constant.